RATIONAL NUMBERS

- A **rational** number is any number which can be expressed as a fraction $\frac{a}{b}$, where a and b are whole numbers (integers).

Example

$\frac{1}{4}, \frac{5}{8}, 6 \left(\frac{6}{1}\right)$ are all rational numbers.

- Some **decimal** numbers can also be written as fractions

Examples

$\frac{1}{4} = 0.25$

$\frac{5}{8} = 0.625$

TERMINATING DECIMALS

$\frac{5}{6} = 0.8333... = 0.8\dot{3}$ **RECURRING DECIMAL**

- All **terminating** decimals are rational because they can be written as a fraction.

Example

$0.625 = \frac{625}{1000} = \frac{125}{200} = \frac{5}{8}$

- All **recurring** decimals are rational because they can be written as a fraction **(with a little work!)**

Example 1

$0.8\dot{3} = 0.8333333...$

$$\begin{aligned} \text{Let } x &= 0.8\dot{3} \\ x \times 100 &= 83.3333... \\ x \times 10 &= 8.3333... \\ 100x - 10x &= 83.3333... - 8.3333... \\ 90x &= 75 \\ x &= \frac{75}{90} = \frac{5}{6} \end{aligned}$$

Notice the technique,

- $0.8\dot{3}$ is first multiplied by 100.
- $0.8\dot{3}$ is then multiplied by 10.
- This has the effect of making the decimal parts of each, the same.
- Subtracting removes the decimal parts.
- A little thought is needed to make sure that the correct multiple of 10 is used.

Example 2

$0.\dot{5}\dot{4} = 0.545454...$

$$\begin{aligned} \text{let } x &= 0.\dot{5}\dot{4} \\ 100x &= 54.545454... \\ 1x &= 0.545454... \\ 100x - 1x &= 54.545454... - 0.545454... \\ 99x &= 54 \\ x &= \frac{54}{99} = \frac{6}{11} \end{aligned}$$

Example 3

$0.34\dot{1}2\dot{6} = 0.34126126...$

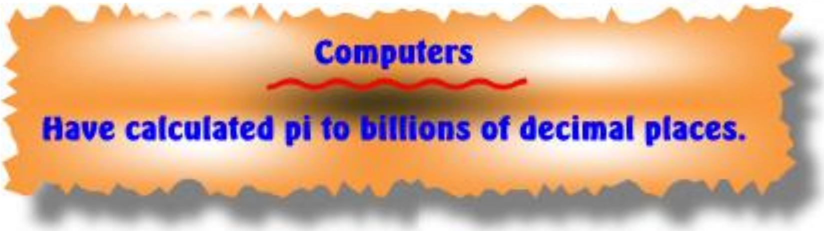
$$\begin{aligned} \text{let } x &= 0.34\dot{1}2\dot{6} \\ 100000x &= 34126.126... \\ 100x &= 34.126... \\ 100000x - 100x &= 34092 \\ 99900x &= 34092 \\ x &= \frac{34092}{99900} = \frac{947}{2775} \end{aligned}$$

IRRATIONAL NUMBERS

- An irrational number is one which is not rational.
- It cannot be written as a fraction in the form $\frac{a}{b}$.
- The decimal form of an irrational number would go on for ever without repeating.
- The most famous irrational number is π.
- Any multiple of π is irrational,

  Example

  2π, 3π, $\pi + 1$ etc. are irrational



- The square root of any number, other than a square number, is irrational.

  $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ etc. are irrational numbers

- Be careful with square roots, $2\sqrt{2}$ is irrational but $\sqrt{2} \times \sqrt{2}$ is rational because $\sqrt{2} \times \sqrt{2} = 2$.

Example 1

Categorise these numbers.

Number	Rational or Irrational	Reason
3π	Irrational	π is irrational
$\sqrt{9}$	Rational	$\sqrt{9} = 3$
$\frac{1}{3}$	Rational	This is a fraction
$\sqrt{10}$	Irrational	10 is not a square number
0.99	Rational	Terminating decimal
$4.\dot{3}$	Rational	Recurring decimal
$1+\sqrt{3}$	Irrational	$\sqrt{3}$ is irrational

Example 2

a) Give an example of an irrational number between 1 and 4.

b) Give an example of an irrational number between 10 and 15.

Solution

a) $\pi = 3.14$ (2 d.p.)
$\therefore$ π is an irrational number between 1 and 4

There are other answers, for example $1^2 = 1$ and $4^2 = 16$ so the square root of any non square number between 1 and 16 is irrational. The following are possible answers: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, … $\sqrt{15}$.

$\sqrt{4}$ is not included since $\sqrt{4} = 2$, and 2 is rational.

b) $\pi + 10$ is an irrational number between 10 and 15.

There are other answers, $\pi + 11$, 2π, 3π etc.

Square roots can also be used, for example $10^2 = 100$ and $15^2 = 225$ so the square root of any non square number between 100 and 225 is irrational. The following are possible answers: $\sqrt{101}$, $\sqrt{102}$, $\sqrt{103}$, … $\sqrt{120}$, $\sqrt{122}$, etc.

$\sqrt{121}$ is not included since $\sqrt{121} = 11$, and 11 is rational.

Example 3

Write down a value of x which is greater than 0 and less than 1 for which 16^x is rational.

Solution

Using a little knowledge about indices, $a^{½} = \sqrt{a}$, in this case, $16^{½} = \sqrt{16} = 4$

$\therefore x = ½$ and $16^{½} = 4$ which is rational.

Example 4

Write down two irrational numbers, a and b such that a + b is irrational but a – b is rational.

Solution

$a = 2 + \sqrt{2}$

$b = 1 + \sqrt{2}$

$a + b = 3 + 2\sqrt{2}$ This is irrational because $\sqrt{2}$ is irrational.

$a - b = 1$ This is rational.

SURDS

- Irrational numbers such as $\sqrt{2}$, $\frac{4}{\sqrt{3}}$, $1-\sqrt{6}$ which involve square roots are called **SURDS**.
- Surds in their simplest form have no surds in the denominator and the smallest possible whole number in the $\sqrt{\ }$ sign.

Example 1

Simplify a) $\sqrt{20}$ b) $\sqrt{180}$

Solution

a) $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \times \sqrt{5} = 2\sqrt{5}$

b) $\sqrt{180} = \sqrt{9 \times 20} = \sqrt{9 \times 4 \times 5} = \sqrt{9} \times \sqrt{4} \times \sqrt{5} = 3 \times 2 \times \sqrt{5} = 6\sqrt{5}$

Example 2

Simplify a) $\frac{5}{\sqrt{6}}$ b) $\frac{2}{\sqrt{2}}$

Solution **(This technique is known as Rationalising the Denominator)**

a) $\frac{5}{\sqrt{6}} = \frac{5}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$ Multiply top and bottom by $\sqrt{6}$

$= \frac{5\sqrt{6}}{6}$ Remember that $\sqrt{6} \times \sqrt{6} = 6$

b) $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ Multiply top and bottom by $\sqrt{2}$

$= \frac{2\sqrt{2}}{2}$ Remember that $\sqrt{2} \times \sqrt{2} = 2$

$= \sqrt{2}$

Example 3

Simplify $(1 + \sqrt{2})(2 + \sqrt{2})$

Solution

$(1 + \sqrt{2})(2 + \sqrt{2}) = 2 + \sqrt{2} + 2\sqrt{2} + \sqrt{2}\sqrt{2}$

$= 2 + 3\sqrt{2} + 2$

$= 4 + 3\sqrt{2}$

Example 4

Simplify a) $\sqrt{\frac{3}{5}}$ b) $\frac{1 + \sqrt{2}}{\sqrt{2}}$

Solution

a) $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$

b) $\frac{1 + \sqrt{2}}{\sqrt{2}} = \frac{1 + \sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} + 2}{2}$